JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 33, No. 1, January–February 2010

Engineering Notes

Hypersphere Stereographic Orientation Parameters

Jeff Mullen* and Hanspeter Schaub†
University of Colorado at Boulder, Boulder, Colorado 80303

DOI: 10.2514/1.46783

I. Introduction

EULER parameters (EPs), also referred to as quaternions, are a nonsingular set of four attitude coordinates that are constrained to a unit norm. The first analytical mapping from EPs to modified Rodrigues parameters (MRPs) is performed by Wiener in his 1962 dissertation [1], in which he discovered a singularity at the 360 deg rotation. In [2], Marandi and Modi exploit the nonuniqueness property of the MRPs by formulating a nonsingular minimal attitude description. Shuster also mentions the MRPs in his well-known survey of attitude parameterizations, and he gives the parameters the name modified Rodrigues parameters [3]. Tsiotras and Longuski point out that the MRPs can be viewed as the result of a stereographic projection of the EP constraint unit hypersphere onto a three-dimensional (3-D) projection hyperplane [4,5]. He also discovers that the natural logarithm function forms an elegant attitude cost (Lyapunov) function in terms of MRPs, which leads to linear MRP feedback with nonlinear stability.

Schaub and Junkins [6] further develop this work by showing that the MRP stereographic projection description discovered by Tsiotras can be expanded to describe general families of attitude parameters called the stereographic orientation parameters (SOPs). In particular, [6] presents the subgroup of symmetric SOPs and shows that the MRPs and classical Rodrigues parameters are a subset of this family. Later on, Southward et. al. [7] develop the full kinematic properties of the symmetric SOPs by allowing the projection point to be placed anywhere on the scalar EP coordinate axis within the EP constraint hypersphere. These symmetric SOPs are expressed algebraically in terms of scalar projection point coordinates and yield minimal set attitude coordinates, in which the singularity can occur at any desired orientation within $0\ \text{deg} < \Phi < 360\ \text{deg}$. In contrast, the asymmetric stereographic attitude parameters (ASOPs) of [6] place the projection point at ± 1 along one of the vector EP coordinate axes. This leads to an interesting behavior, in which singularities are only encountered if a pure rotation about a particular principle body axis is performed. Further, a $+180$ deg rotation may lead to a singular attitude description, but a -270 deg rotation (exact same orientation) is nonsingular. Only -630 deg in the negative direction leads to a singular description. The nonsymmetric nature of the singular rotations and their dependency of the path to a particular orientation lead to the name of asymmetric SOPs.

Received 18 August 2009; revision received 28 October 2009; accepted for publication 28 October 2009.

*Graduate Student, Aerospace Engineering Sciences Department. Student Member AIAA.

†Associate Professor, Aerospace Engineering Sciences Department; Colorado Center for Astrodynamics Research. Associate Fellow AIAA.

Other recent attitude coordinates that relate to the MRPs include the higher-order Rodrigues parameters [8]. Here, higher-order Cayley transforms are used to develop attitude coordinates that grow infinitely large a at multiples of 360 deg. These higher-order Rodrigues parameters are convenient to develop minimal sets of attitude coordinates, for which the differential equation can be made arbitrarily linear through the use of higher-order Cayley transformations. Hurtado uses the MRPs to create inner and outer parameters for attitude representations and presents new Cayley-like transformations [9].

This paper investigates a subfamily of attitude coordinates called the hypersphere SOPs (HSOPs), which contain both the previous MRPs (particular set of symmetric SOPs) and the ASOP, allowing for all this work to be combined into a single, minimal attitude parameter description. HSOPs allow the projection point to lie at any point on the EP unit hypersphere constraint. Thus, depending on the choice of the project point, these attitude coordinates can display a singular behavior similar to that of the ASOP. The attitude of a spinning body can be described singularity-free, with a minimal three-parameter coordinate set as long as the body is not spinning about a particular combination of principal body axes. Or, the HSOP coordinates can be chosen, such that their singular behavior matches that of the MRPs, for which a particular 360 deg rotation about any body axis leads to a singular description. When different attitude coordinates are combined into a more general family of parameters, such as the joining of classical Rodrigues parameters and MRPs into symmetric SOPs in [7], the result is often a more complex set of algebraic equations. This paper investigates how a general projection point on the surface of the EP constraint hypersphere complicates the associated HSOP differential kinematic equations and their mapping to the shadow set.

This paper is organized as follows. Section II describes the geometry and algebra of a general stereographic projection. Section III provides the analytical mapping between HSOPs from EPs, the DCM, as well as the derivation of the shadow sets, the kinematic differential equation, and the singularity condition. Section IV discusses how the HSOPs can be employed in attitude control strategies.

II. Generalized Stereographic Projections

The four EP attitude coordinates are expressed as a 4×1 matrix:

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \tag{1}$$

Here, each β_i represents an EP coordinate, and the magnitude of this stack is $\boldsymbol{\beta} \cdot \boldsymbol{\beta} = 1$. In terms of the principal rotation angle Φ and the principal rotation axis $\hat{\mathbf{e}} = (e_1, e_2, e_3)$, the EP coordinates are expressed as $\beta_0 = \cos(\Phi/2)$ and $\beta_i = e_i \sin(\Phi/2)$, with $i = 1, 2, 3$. Geometrically, the EP constraint $\boldsymbol{\beta} \cdot \boldsymbol{\beta} = 1$ defines the surface of a four-dimensional (4-D) unit hypersphere, upon which all EPs must lie [10].

Generalized SOPs are a minimal coordinate representation of a particular orientation. These generalized parameters are obtained by projecting the 4-D EP attitude description ($\boldsymbol{\beta}$) onto a 3-D hyperplane, as illustrated in Fig. 1. The coordinates of this intersection point form the SOPs. This section first presents the general stereographic mapping of the unit constraint surface onto a general projection hyperplane. This section considers the particular mapping, in which the projection point $\mathbf{a}$ is a constraint to the hypersphere surface. The projection point $\mathbf{a}$, for now, is a general location. The projection

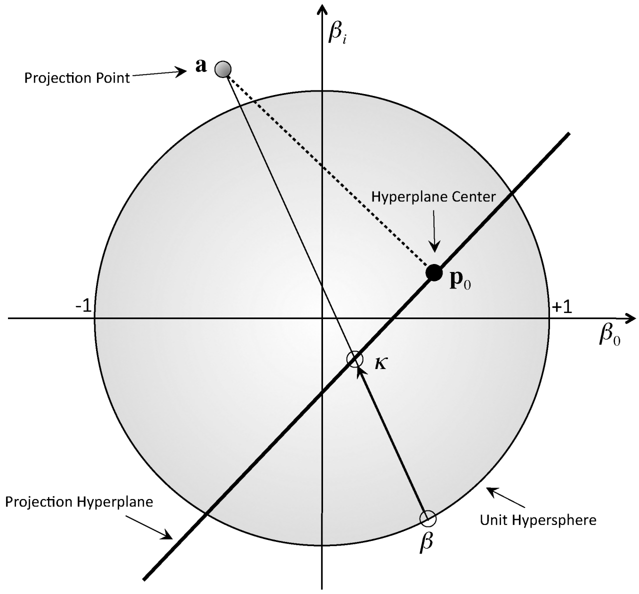

Fig. 1 General stereographic orientation parameter geometry.

hyperplane is defined by the normal of $\mathbf{a}-\mathbf{p_0}$, where $\mathbf{p_0}$ is the closest hyperplane point to $\mathbf{a}$. Therefore, in order to solve for general SOPs, the line formed between the projection point and any EP attitude description is projected onto the hyperplane. The projection of $\boldsymbol{\beta}$ onto the hyperplane is the intersection point $\mathbf{k}$.

The vector equation for the line between the points $\mathbf{a}$ and $\boldsymbol{\beta}$ in n-dimensional space is

$$\mathbf{L} = \mathbf{a} + t(\boldsymbol{\beta} - \mathbf{a}) \tag{2}$$

where t is a free scalar parameter that can describe any point on this line. The generalized equation for a plane in space with the normal vector $\mathbf{a}-\mathbf{p_0}$ that crosses through point $\mathbf{p_0}$ is

$$\mathbf{P} = (\mathbf{a}-\mathbf{p_0})\cdot\boldsymbol{\kappa} = (\mathbf{a}-\mathbf{p_0})\cdot\mathbf{p_0} \tag{3}$$

where $\boldsymbol{\kappa}$ is the intersection point on the hyperplane of the EP description through the projection point. To solve for $\boldsymbol{\kappa}$, the line defined in Eq. (2) is intersected with the plane described by Eq. (3). Solving for the scalar parameter t and substituting into Eq. (2) yields:

$$\boldsymbol{\kappa} = \mathbf{a} + \frac{(\mathbf{a}-\mathbf{p_0})\cdot(\mathbf{p_0}-\mathbf{a})}{(\mathbf{a}-\mathbf{p_0})\cdot(\boldsymbol{\beta}-\mathbf{a})}(\boldsymbol{\beta}-\mathbf{a}) \tag{4}$$

This 4-D intersection point $\boldsymbol{\kappa}$ is used to derive the HSOPs. The rest of the algebra will project this 4-D point to a 3-D hyperplane.

III. Derivation of the Hypersphere Stereographic Orientation Parameters

A. Relationship to Euler Parameters

The HSOP attitude set is formed by two conditions: 1) the projection point $\mathbf{a}$ is constrained to the surface of the EP constraint unit hypersphere, and 2) the projection hyperplane passes through the center of this hypersphere.

With the projection point defined as $\mathbf{a} = [a_0, a_1, a_2, a_3]^T$, this means that $\mathbf{a}\cdot\mathbf{a} = 1$. Likewise, the EP stack will be defined by $\boldsymbol{\beta} = [\beta_0, \beta_1, \beta_2, \beta_3]^T$ and from the EP constraint $\boldsymbol{\beta}\cdot\boldsymbol{\beta} = 1$. Finally, we set $\mathbf{p_0} = [0, 0, 0, 0]^T$ to ensure the projection hyperplane intersects with the origin. This simplifies Eq. (4) to

$$\boldsymbol{\kappa} = \mathbf{a} - \frac{\boldsymbol{\beta}-\mathbf{a}}{\mathbf{a}\cdot\boldsymbol{\beta}-1} \tag{5}$$

The 4-D vector $\boldsymbol{\kappa}$ represents the projection point of the attitude description $\boldsymbol{\beta}$ onto the hyperplane. To be a minimal set, this vector needs to be represented in terms of the three basis vectors that describe the projection hyperplane. The 3-D hyperplane is written in the following vector form: $\mathbf{P} = \zeta_1\mathbf{e_1} + \zeta_2\mathbf{e_2} + \zeta_3\mathbf{e_3}$. Here, ζ_1, ζ_2, and ζ_3 are the in-plane coordinates of the intersection, and thus the HSOP coordinates. The vectors $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$ form the base vectors of the hyperplane by being mutually orthogonal to the projection point vector $\mathbf{a}$. The projection point can be expressed as

$$\boldsymbol{\kappa} = [A]\begin{bmatrix}\boldsymbol{\zeta}\\ 0\end{bmatrix} \tag{6}$$

where $\boldsymbol{\zeta}$ is the 3×1 matrix:

$$\boldsymbol{\zeta} = \begin{bmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{bmatrix} \tag{7}$$

and $[A]$ is

$$[A] = \begin{bmatrix}\vdots & \vdots & \vdots & \vdots\\ \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} & \mathbf{a}\\ \vdots & \vdots & \vdots & \vdots\end{bmatrix} \tag{8}$$

Note that the base vector $\mathbf{e_i}$ are not unique. For example, the base vector can always be rotated about $\mathbf{a}$ to form a new set. This would result in alternate numerical HSOP descriptions, which describe the same orientation. However, these alternate formulations have identical singular behaviors and do not provide any practical benefits. A goal of this paper is to determine a set of base vectors $\mathbf{e_i}$, such that the resulting HSOPs are identical to the prior MRP or ASOP coordinates if the proper projection point is chosen.

For the developments of this paper, a unit-length orthogonal basis will be used to describe the projection plane. Therefore, this matrix is full rank and orthogonal, meaning that $[A]^{-1} = [A]^T$. Solving for the matrix of HSOP coordinates, $\boldsymbol{\zeta}$ results in

$$\begin{bmatrix}\boldsymbol{\zeta}\\ 0\end{bmatrix} = [A]^T\left[\mathbf{a} - \frac{\boldsymbol{\beta}-\mathbf{a}}{\mathbf{a}\cdot\boldsymbol{\beta}-1}\right] \tag{9}$$

To generate $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$, the composite rotation property of EPs [10] is used. Because $\mathbf{a}$ is constrained to the unit hypersphere, it can be treated as a valid EP attitude description. Therefore, an orthogonal 4-D basis of EPs ($\mathbf{e_i}$) is created by simply adding rotations of -180 deg to $\mathbf{a}$. This results in the following set of unit-length basis vectors:

$$\begin{aligned}\mathbf{e_1} &= [a_1, -a_0, -a_3, a_2]^T\\ \mathbf{e_2} &= [a_2, a_3, -a_0, -a_1]^T\\ \mathbf{e_3} &= [a_3, -a_2, a_1, -a_0]^T\end{aligned} \tag{10}$$

Substituting these vectors into Eq. (9) and expanding results in the minimal HSOP attitude description:

$$\boldsymbol{\zeta} = \frac{1}{1-\mathbf{a}\cdot\boldsymbol{\beta}}\begin{bmatrix}a_1\beta_0 - a_0\beta_1 - a_3\beta_2 + a_2\beta_3\\ a_2\beta_0 + a_3\beta_1 - a_0\beta_2 - a_1\beta_3\\ a_3\beta_0 - a_2\beta_1 + a_1\beta_2 - a_0\beta_3\end{bmatrix} \tag{11}$$

It is important to note that the $\boldsymbol{\zeta} = \mathbf{0}$ vector does not correspond to the zero attitude of $\beta_0 = 1$. Instead, the HSOP representation of the zero orientation is defined as

$$\boldsymbol{\zeta}_{\mathbf{0}} = \frac{1}{1-a_0}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix} \tag{12}$$

By using this definition, it is clear that the HSOPs are simply a rotation of the MRPs by the attitude described by the point $\mathbf{a}$; however, the zero attitude is directly dependent on the parameter $\mathbf{a}$. This has implications on the singularity condition as well as the attitude control, which will be discussed in further detail in the following sections.

To verify these HSOP definitions are a family of attitude coordinates containing MRPs and ASOPs, we perform the following checks. Reducing Eq. (11) to the MRP case, where $\mathbf{a}$ lies at $\beta_0 = -1$, results in the definition of the MRPs [10]:

$$\begin{bmatrix} \zeta_1 = \sigma_1 \\ \zeta_2 = \sigma_2 \\ \zeta_3 = \sigma_3 \end{bmatrix} = \begin{bmatrix} \frac{\beta_1}{1+\beta_0} \\ \frac{\beta_2}{1+\beta_0} \\ \frac{\beta_3}{1+\beta_0} \end{bmatrix} \tag{13}$$

Reducing Eq. (11) to the ASOP case, where $\mathbf{a}$ lies at $\beta_1 = -1$, results in

$$\begin{bmatrix} \zeta_1 = \eta_1 \\ \zeta_2 = \eta_2 \\ \zeta_3 = \eta_3 \end{bmatrix} = \begin{bmatrix} \frac{-\beta_0}{1+\beta_1} \\ \frac{\beta_3}{1+\beta_1} \\ \frac{-\beta_2}{1+\beta_1} \end{bmatrix} \tag{14}$$

The original ASOP case (as presented in [6,10]) has a sign discrepancy on the first and third parameters, and the second and third parameters are switched. This is due to the uniqueness issue when defining the basis of the projection plane. The orientation of the original ASOP projection hyperplane was chosen at random without any further information. The presented ASOP hyperplane definition is preferred, because it allows the presented base vector $\mathbf{e}_i$ definition to map between EP and HSOP in a general way. Clearly, there is no practical difference between the ζ_i and η_i coordinates. Further advantages of this convention become apparent when considering the HSOP differential kinematic equations.

Solving for the inverse transformation from $\boldsymbol{\zeta}$ to $\boldsymbol{\beta}$ is similar to solving for the intersection point on the projection hyperplane. However, rather than solving for the intersection of a plane, the reverse will happen, and the intersection of the projection line onto the unit hypersphere will be accomplished.

Assume vectors $\mathbf{a}$ and $\boldsymbol{\kappa}$ (the projection point and intersection point, respectively) are known. Therefore, Eq. (2) can be written in the equivalent form $\mathbf{L} = \mathbf{a} + u(\boldsymbol{\kappa} - \mathbf{a})$, where u is a scalar parameter of the line. Because $\mathbf{a}$ and $\boldsymbol{\beta}$ lie on the unit hypersphere, intersecting this line with the unit hypersphere results in two intersection points: one at $\mathbf{a}$ and one at $\boldsymbol{\beta}$. The equation of an n-dimensional unit hypersphere in space is given as $|\mathbf{S}|^2 = 1$. Intersecting both of these equations results in the following vector equation:

$$|\mathbf{a} + u(\boldsymbol{\kappa} - \mathbf{a})|^2 = 1 \tag{15}$$

Solving for the parameter u and applying the unit hypersphere constraints results in the following:

$$u = \frac{1 \pm 1}{1 + \zeta^2} \tag{16}$$

where $\zeta^2 = \boldsymbol{\zeta}^T \boldsymbol{\zeta} = \zeta_1^2 + \zeta_2^2 + \zeta_3^2$. Because of the unit-length constraints for the projection point vector and the EP description, this means there are two solutions, because both of these points lie on the unit hypersphere. Substituting the nontrivial solution back into the general equation for a line results in the desired EP description vector $\boldsymbol{\beta}$. The inverse relation between the HSOPs and EPs is shown next:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} a_0 + \frac{2}{1+\zeta^2}(a_1\zeta_1 + a_2\zeta_2 + a_3\zeta_3 - a_0) \\ a_1 + \frac{2}{1+\zeta^2}(-a_0\zeta_1 + a_3\zeta_2 - a_2\zeta_3 - a_1) \\ a_2 + \frac{2}{1+\zeta^2}(-a_3\zeta_1 - a_0\zeta_2 + a_1\zeta_3 - a_2) \\ a_3 + \frac{2}{1+\zeta^2}(a_2\zeta_1 - a_1\zeta_2 - a_0\zeta_3 - a_3) \end{bmatrix} \tag{17}$$

In the simple compact matrix form,

$$\boldsymbol{\beta} = \mathbf{a} + \frac{2}{1 + \zeta^2}([A]\boldsymbol{\zeta} - \mathbf{a}) \tag{18}$$

To verify this general inverse mapping from EP to HSOP, let us consider the special cases of mapping from EPs to the previously developed MRPs or ASOPs. Calculating the inverse relation for the MRP case ($\mathbf{a} = [-1, 0, 0, 0]^T$) and switching coordinates from ζ to σ results in the definition of the inverse for MRPs [10]:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \frac{1}{1 + \sigma^2} \begin{bmatrix} 1 - \sigma^2 \\ 2\sigma_1 \\ 2\sigma_2 \\ 2\sigma_3 \end{bmatrix} \tag{19}$$

Calculating the inverse relation for the ASOP case ($\mathbf{a} = [0, -1, 0, 0]^T$) results in the definition of the inverse relation for ASOPs:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \frac{1}{1 + \eta^2} \begin{bmatrix} -2\eta_1 \\ 1 - \eta^2 \\ -2\eta_3 \\ 2\eta_2 \end{bmatrix} \tag{20}$$

B. Direction Cosine Matrix Derivation

Because the HSOPs were derived from the EP set, the direction cosine matrix (DCM) definition for the EP is used:

$$[C] = [I_{3\times3}] - 2\beta_0[\tilde{\bar{\boldsymbol{\beta}}}] + 2[\tilde{\bar{\boldsymbol{\beta}}}][\tilde{\bar{\boldsymbol{\beta}}}] \tag{21}$$

where $\bar{\boldsymbol{\beta}}$ is the vectorial component of the EPs and $[\tilde{\bar{\boldsymbol{\beta}}}]$ is the cross-product matrix defined as

$$[\tilde{\bar{\boldsymbol{\beta}}}] = \begin{bmatrix} 0 & -\beta_3 & \beta_2 \\ \beta_3 & 0 & -\beta_1 \\ -\beta_2 & \beta_1 & 0 \end{bmatrix} \tag{22}$$

Define $\bar{\mathbf{a}}$ to be

$$\bar{\mathbf{a}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{23}$$

Substituting Eq. (17) into Eq. (22) results in the following expression:

$$[\tilde{\bar{\boldsymbol{\beta}}}] = \frac{2}{1 + \zeta^2}\left[\frac{\zeta^2 - 1}{2}[I_{3\times3}] + [\tilde{\boldsymbol{\zeta}}]\right][\tilde{\bar{\mathbf{a}}}] - \frac{2}{1 + \zeta^2}(a_0[I_{3\times3}] + [\tilde{\bar{\mathbf{a}}}])[\tilde{\boldsymbol{\zeta}}] \tag{24}$$

Simplifying β_0 results in

$$\beta_0 = a_0 \frac{\zeta^2 - 1}{\zeta^2 + 1} + \frac{2}{\zeta^2 + 1}\bar{\mathbf{a}}^T \boldsymbol{\zeta} \tag{25}$$

Reducing this DCM down to the MRP case, where $\mathbf{a} = [-1, 0, 0, 0]^T$, and performing a variable switch from ζ to σ results in the same DCM as for the MRPs [10]:

$$[C]_{\mathrm{MRP}} = \frac{1}{(1+\sigma^2)^2}\begin{bmatrix} 4(\sigma_1^2-\sigma_2^2-\sigma_3^2)+\Sigma_\sigma^2 & 8\sigma_1\sigma_2+4\sigma_3\Sigma_\sigma & 8\sigma_1\sigma_3-4\sigma_2\Sigma_\sigma \\ 8\sigma_2\sigma_1-4\sigma_3\Sigma_\sigma & 4(-\sigma_1^2+\sigma_2^2-\sigma_3^2)+\Sigma_\sigma^2 & 8\sigma_2\sigma_3+4\sigma_1\Sigma_\sigma \\ 8\sigma_3\sigma_1+4\sigma_2\Sigma_\sigma & 8\sigma_3\sigma_2-4\sigma_1\Sigma_\sigma & 4(-\sigma_1^2-\sigma_2^2+\sigma_3^2)+\Sigma_\sigma^2 \end{bmatrix} \tag{26}$$

where $\Sigma_\sigma = (1-\sigma^2)$. Reducing the HSOP DCM to the ASOP case, where $\mathbf{a} = [0, \ -1, \ 0, \ 0]^T$, and switching the parameters from ζ to η results in a similar DCM as seen in Eq. (26):

$$[C]_{\mathrm{ASOP}} = \frac{1}{(1+\eta^2)^2}\begin{bmatrix} 4(\eta_1^2-\eta_2^2-\eta_3^2)+\Sigma_\eta^2 & -8\eta_1\eta_2-4\eta_3\Sigma_\eta & -8\eta_1\eta_3+4\eta_2\Sigma_\eta \\ 8\eta_2\eta_1-4\eta_3\Sigma_\eta & 4(\eta_1^2-\eta_2^2+\eta_3^2)-\Sigma_\eta^2 & -8\eta_2\eta_3-4\eta_1\Sigma_\eta \\ 8\eta_3\eta_1+4\eta_2\Sigma_\eta & -8\eta_3\eta_2+4\eta_1\Sigma_\eta & 4(\eta_1^2+\eta_2^2-\eta_3^2)-\Sigma_\eta^2 \end{bmatrix} \tag{27}$$

with $\Sigma_\eta = (1-\eta^2)$.

C. Hypersphere Stereographic Orientation Parameter Shadow Set Derivation

Because EPs have four parameters, there is redundancy in the attitude description (a single EP is not unique). The other EPs that describes the same attitude are known as the shadow set $\boldsymbol{\beta}^s = -\boldsymbol{\beta}$. The EP shadow set simply represents another way to rotate the object to the desired attitude. For example, if the desired attitude is 45 deg about the $\mathbf{e}_1$ body axis, one could either rotate 45 deg (short rotation) or -315 deg (long rotation) about the body $\mathbf{e}_1$ axis. Both rotations describe the same orientation. However, there are two different EPs that describe both a short and a long rotation. Because of this shadow parameter with EPs, there is also a shadow set of HSOPs. The geometry of this is illustrated in Fig. 2.

Substituting the definition of the EP shadow set into Eq. (6) results in the HSOP shadow set solution:

$$\boldsymbol{\zeta}^s = [A]^T\left[\mathbf{a} - \frac{\boldsymbol{\beta}+\mathbf{a}}{\mathbf{a}\cdot\boldsymbol{\beta}+1}\right] \tag{28}$$

Expanding this form results in the explicit HSOP shadow set definition:

$$\boldsymbol{\zeta}^s = \frac{1}{\mathbf{a}\cdot\boldsymbol{\beta}-1}\begin{bmatrix} -a_1\beta_0+a_0\beta_1+a_3\beta_2-a_2\beta_3 \\ -a_2\beta_0-a_3\beta_1+a_0\beta_2+a_1\beta_3 \\ -a_3\beta_0+a_2\beta_1-a_1\beta_2+a_0\beta_3 \end{bmatrix} \tag{29}$$

$$\boldsymbol{\zeta}^s = \frac{1}{\mathbf{a}\cdot\boldsymbol{\beta}-1}(-\beta_0\bar{\mathbf{a}} + a_0[I_{3\times3}] + [\tilde{\bar{\mathbf{a}}}]\bar{\boldsymbol{\beta}}) \tag{30}$$

Substituting in the inverse relation between the EPs and the HSOPs [Eq. (17)], and simplifying, results in the elegant shadow set relation:

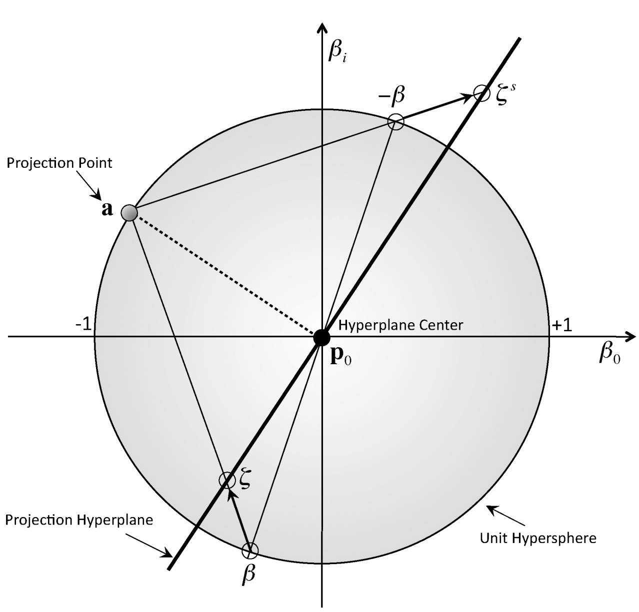

Fig. 2 HSOP shadow set geometry.

$$\zeta_i^s = -(\zeta_1, \ \zeta_2, \ \zeta_3)/|\boldsymbol{\zeta}|^2 \tag{31}$$

This is the same shadow set algebraic relationship as found in the MRP shadow set transformation. This result is expected, as the HSOPs (geometrically) can be represented as a rotation of the MRPs about a particular attitude. The HSOP shadow set, much like the shadow set of the MRP cases, can be used to avoid singularities during integration. As shown in Fig. 2, as one parameter nears the projection point, the HSOP description begins to grow infinitely large, whereas the shadow parameter stays well bounded. This switching could, however, occur at any point before a singularity is reached, as the shadow set describes the same attitude as the normal set. A property about the shadow sets of the HSOPs is when one set magnitude is greater than one and the shadow set is always less than one. When one set has a magnitude of one, the other also has a magnitude of one. Switching at a magnitude of one is somewhat arbitrary and can happen at any time, however, when $|\boldsymbol{\sigma}| = 1$ always corresponds to a 180 deg rotation from the null attitude. Likewise, when the HSOPs are at $|\boldsymbol{\zeta}| = 1$, it corresponds to the 180 deg rotation from the attitude specified by the parameter $\mathbf{a}$. Switching both sets at a magnitude of one keeps the parameters well bounded by maintaining an attitude description that is always less than 180 deg and avoids singularities.

D. Kinematic Differential Equation

The derivation of the HSOP differential kinematic equation requires the kinematic differential equation of EPs [10]:

$$\dot{\boldsymbol{\beta}} = \frac{1}{2}\begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix}\begin{bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{32}$$

Differentiating Eq. (11) results in

$$\dot{\zeta}_1 = \frac{-a_1\dot{\beta}_0+a_0\dot{\beta}_1+a_3\dot{\beta}_2-a_2\dot{\beta}_3}{a_0\beta_0+a_1\beta_1+a_2\beta_2+a_3\beta_3-1} + \frac{(a_1\beta_0-a_0\beta_1-a_3\beta_2+a_2\beta_3)(a_0\dot{\beta}_0+a_1\dot{\beta}_1+a_2\dot{\beta}_2+a_3\dot{\beta}_3)}{(a_0\beta_0+a_1\beta_1+a_2\beta_2+a_3\beta_3-1)^2} \tag{33a}$$

$$\dot{\zeta}_2 = \frac{-a_2\dot{\beta}_0-a_3\dot{\beta}_1+a_0\dot{\beta}_2+a_1\dot{\beta}_3}{a_0\beta_0+a_1\beta_1+a_2\beta_2+a_3\beta_3-1} + \frac{(a_2\beta_0+a_3\beta_1-a_0\beta_2-a_1\beta_3)(a_0\dot{\beta}_0+a_1\dot{\beta}_1+a_2\dot{\beta}_2+a_3\dot{\beta}_3)}{(a_0\beta_0+a_1\beta_1+a_2\beta_2+a_3\beta_3-1)^2} \tag{33b}$$

$$\dot{\zeta}_2 = \frac{-a_3\dot{\beta}_0+a_2\dot{\beta}_1-a_1\dot{\beta}_2+a_0\dot{\beta}_3}{a_0\beta_0+a_1\beta_1+a_2\beta_2+a_3\beta_3-1} + \frac{(a_3\beta_0-a_2\beta_1+a_1\beta_2-a_3\beta_3)(a_0\dot{\beta}_0+a_1\dot{\beta}_1+a_2\dot{\beta}_2+a_3\dot{\beta}_3)}{(a_0\beta_0+a_1\beta_1+a_2\beta_2+a_3\beta_3-1)^2} \tag{33c}$$

Substituting in Eqs. (17) and (32), and applying the unit hypersphere constraint, results in the kinematic differential equation for the HSOPs, which are independent of the projection point **a**. Despite the generalities with this attitude parameter set, the result is extremely elegant and given in the following matrix form:

$$\dot{\boldsymbol{\zeta}} = \frac{1}{4}\begin{bmatrix} 1-\zeta^2+2\zeta_1^2 & 2(\zeta_1\zeta_2-\zeta_3) & 2(\zeta_1\zeta_3+\zeta_2) \\ 2(\zeta_2\zeta_1+\zeta_3) & 1-\zeta^2+2\zeta_2^2 & 2(\zeta_2\zeta_3-\zeta_1) \\ 2(\zeta_3\zeta_1-\zeta_2) & 2(\zeta_3\zeta_2+\zeta_1) & 1-\zeta^2+2\zeta_3^2 \end{bmatrix}\boldsymbol{\omega} \quad (34)$$

Because of the lack of information about **a** in these equations, we find that all HSOP differential kinematic equations will have the exact same algebraic form. These equations are written in compact vector notation as

$$\dot{\boldsymbol{\zeta}} = \tfrac{1}{4}[(1-\zeta^2)[I_{3\times 3}] + 2[\tilde{\zeta}] + 2\boldsymbol{\zeta}\boldsymbol{\zeta}^T]\boldsymbol{\omega} \quad (35)$$

After applying the coordinate switch from ζ to σ, this kinematic differential equation is the same as for the MRPs [10]. The only difference between the HSOPs and the MRPs is that the HSOPs are simply a rotation of the MRPs by the attitude defined by **a**. Therefore, the kinematic differential equations will be the same. With this new ASOP definition, the ASOP differential kinematic equations assume the identical algebraic form as the MRPs. This elegant result justifies why the ASOPs should be redefined, as shown in this paper.

E. Singularity Condition

Because an HSOP is a minimal attitude coordinate set, there is a singular description that can be arbitrarily placed. This condition arises when the denominator in Eq. (11) equals zero:

$$1 - a_0\beta_0 - a_1\beta_1 - a_2\beta_2 - a_3\beta_3 = 0 \quad (36)$$

Geometrically, this arises at one point on the unit hypersphere when $\mathbf{a} = \boldsymbol{\beta}$. Writing this in terms of the principal rotation angle Φ and the principal axis components e_i results in

$$a_0 = \cos(\Phi/2) = \beta_0; \qquad a_1 = e_1\sin(\Phi/2) = \beta_1$$
$$a_2 = e_2\sin(\Phi/2) = \beta_2; \qquad a_3 = e_3\sin(\Phi/2) = \beta_3 \quad (37)$$

Therefore, from Eq. (37), the singularity can be placed in any desired direction with any desired rotation angle. There are two rotations that can be performed in order to reach a singularity. For example, if the singularity is placed in a certain direction at $\Phi = 135$ deg, a rotation of $+135$ or -585 deg about the appropriate direction could be performed in order to reach the singular point. This means that with the HSOP and its subsets, the rotational path to a particular orientation determines if the attitude description will go singular.

It is important to note that, because this singular point can be placed anywhere on the unit hypersphere, there is only one attitude description that it can go singular at, and in order for this to be a singularity, it must lie exactly on **a**. However, attitudes around this point will grow infinitely large and, depending on the control technique applied, these large values can be used to avoid a neighborhood of points in space.

IV. Spacecraft Control

Because the HSOP differential kinematic equations are algebraically equivalent to the MRP differential kinematic equations, any control development that exploits algebraic properties of the MRP differential equations [5,6,11,12] can also be directly applied to the HSOP. This process is illustrated in the following control development example, based on the MRP-based attitude control developed in [6]. For the purposes of this discussion, the underlying rotational dynamics will be Euler's equations of rotational motion of a rigid body:

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{u} + \boldsymbol{L} \quad (38)$$

where $[I]$ is the inertia tensor, $\boldsymbol{u}$ is the control torque, and $\boldsymbol{L}$ is any external torque acting on the body. As developed for the MRP set, a logarithmic Lyapunov function will be used to design a stabilizing control law [5,6]:

$$V(\delta\boldsymbol{\omega}, \boldsymbol{\zeta}_e) = \tfrac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega} + 2K\ln(1+\boldsymbol{\zeta}_e^2) \quad (39)$$

where $\boldsymbol{\zeta}_e$ is defined as the error attitude from the zero attitude of the body, so that $\boldsymbol{\zeta}_e = \boldsymbol{\zeta} - \boldsymbol{\zeta}_0$, where $\boldsymbol{\zeta}_0$ is defined in Eq. (12). This is required, because a zero HSOP attitude does not correspond to the null attitude of a body; by adding in this offset factor, as $\boldsymbol{\zeta}_e$ is driven to zero, the attitude of the body is driven to the null attitude. To ensure global stability, the Lyapunov rate is set equal to the negative semidefinite $\dot{V} = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega}$. Because $\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$, we can plug in the equations of motion in order to generate the closed-loop dynamics and solve for the control variable. This leads to the stabilizing control law $\boldsymbol{u}$ [6]:

$$\boldsymbol{u} = [\tilde{\omega}][I]\boldsymbol{\omega} + [I](\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}_r) - K\boldsymbol{\zeta}_e - [P]\delta\boldsymbol{\omega} - \boldsymbol{L} \quad (40)$$

Although this attitude control law has the exact same algebraic form as the MRP-based control law, the closed-loop response will be different, because the HSOP represents a different attitude description in general, and has a completely different singular behavior.

V. Conclusions

The newly developed hypersphere SOPs are a generalization of the MRPs and ASOPs. Direct analytical mappings are present from the HSOP to the EPs, as well as the DCM. The key result of this paper is that the HSOPs have the same differential kinematic equation as the MRPs, and thus can be applied to any control law that uses the algebraic form of the MRP kinematic equation.

HSOPs are different than MRPs because of the different singular behaviors of each attitude coordinate set. This offers great flexibility, as the singular orientation can be placed at a full revolution or at particular rotations about particular body axes. In all cases, the kinematic differential equation only has quadratic nonlinear terms equivalent to those of the MRPs. This is highly beneficial in relation to nonlinear spacecraft attitude control, as HSOPs will have the same stability guarantees as a controller designed using MRPs.

Acknowledgment

The authors would like thank Rajtilok Chakravarty for his support in the early efforts of exploring the idea of HSOPs.

References

[1] Wiener, T. F., "Theoretical Analysis of Gimballess Inertial Reference Equipment Using Delta-Modulated Instruments, Ph.D. Thesis, Massachusetts Inst. of Technology, Cambridge, MA, 1962.

[2] Marandi, S. R., and Modi, V. J., "A Preferred Coordinate System and the Associated Orientation Representation in Attitude Dynamics," *Acta Astronautica*, Vol. 15, No. 11, 1987, pp. 833–843. doi:10.1016/0094-5765(87)90038-5

[3] Shuster, M. D., "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993, pp. 439–517.

[4] Tsiotras, P., and Longuski, J. M., "A New Parameterization of the Attitude Kinematics," *Journal of the Astronautical Sciences*, Vol. 43, No. 3, 1996, pp. 342–262.

[5] Tsiotras, P., "New Control Laws for the Attitude Stabilization of Rigid Bodies," *13th IFAC Symposium on Automatic Control in Aerospace*, International Federation of Automatic Control, Laxenburg, Austria, Sept. 1994, pp. 316–321.

[6] Schaub, H., and Junkins, J. L., "Stereographic Orientation Parameters for Attitude Dynamics: A Generalization of the Rodrigues Parameters," *Journal of the Astronautical Sciences*, Vol. 44, No. 1, 1996, pp. 13–15.

[7] Southward, C. M., Ellis, J. R., and Schaub, H., "Symmetric Stereographic Orientation Parameters Applied to Constrained Spacecraft Attitude Control," *Journal of the Astronautical Sciences*, Vol. 55, No. 3, July–Sept 2007, pp. 389–405.

[8] Tsiotras, P., Junkins, J. L., and Schaub, H., "Higher Order Cayley Transforms with Applications to Attitude Representations," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 3, May–June 1997,

pp. 528–534.
doi:10.2514/2.4072

[9] Hurtado, J. E., “Interior Parameters, Exterior Parameters, and a Cayley-Like Transform,” *Journal of Guidance, Control, and Dynamics*, Vol. 32, No. 2, 2009, pp. 653–657.
doi:10.2514/1.39624

[10] Schaub, H., and Junkins, J. L., *Analytical Mechanics of Space Systems*, AIAA Education Series, AIAA, Reston, VA, Oct. 2003.

[11] Yoon, H., and Tsiotras, P., “Spacecraft Adaptive Attitude and Power Tracking with Variable Speed Control Moment Gyroscopes,” *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 6, Nov.–Dec. 2002, pp. 1081–1090.
doi:10.2514/2.4987

[12] Schaub, H., Akella, M., and Junkins, J. L., “Adaptive Control of Nonlinear Attitude Motions Realizing Linear Closed Loop Dynamics,” *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 1, Jan.–Feb. 2001, pp. 95–100.
doi:10.2514/2.4680